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ON THE RELATION BETWEEN A TYPE THEORETIC AND A LOGICAL FORMULATION OF THE THEORY OF CONSTRUCTIONS

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Avhandling

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1. INTRODUCTION.

We will present two Theories of Constructions, a Type theoretic (TTC) and a Logical one (LTC). The theory TTC is based on the type theoretic approach made by Martin-Löf [4] and the theory LTC on ordinary predicate logic. Which one of these theories that is taken as most fundamental will depend on ones views on the foundations of constructive mathematics. We will show, however, that if one of the theories is chosen, there is a natural way of interpreting the other.

To every formula A of LTC we will associate a type A^o, namely the type of constructions which prove the formula A. The construction of A^o from A will correspond to the interpretation of the logical constants given by Heyting [2]. This way of interpreting the logical constants is also connected with Kleene's realizability interpretation [3], in which the notion of recursive function replaces the general notion of constructive function. We will then show that, if A is derivable in LTC, then we can construct an expression a^o and give a derivation of $a^o \in A^o$ in TTC. This will give the interpretation of LTC in TTC.

The interpretation of TTC in LTC will consist of associating with each type A a formula $A^*[z]$, such that if $a \in A$ is derivable in TTC, then $A^*[a]$ is derivable in LTC. $A^*[z]$ will be constructed from A by expressing in LTC the condition

for an object to be an element of A.

I am very grateful to my thesis adviser Per Martin-Löf for his kind guidance of my work.

2. TYPE THEORETIC FORMULATION OF THE THEORY OF CONSTRUCTIONS (TTC).

The theory TTC is a variant of the theory of types presented by Martin-Löf [4]. It consists of rules of expression and type formation, axioms and rules of inference. There are two relations, the epsilon relation $\in$ and the definitional equality =. The rules of inference will be formulated in natural deduction in the spirit of Gentzen [1].

If x is a variable which may occur free in an expression b, we will sometimes denote b by b[x] and let b[a] denote the expression we get by substituting another expression a for all free occurances of x in b. The same notation will be used for types. The notation $[x \in A]$ in a derivation is used to indicate that the assumption $x \in A$ may be discharged.

Rules of expression formation.

1. Variables x, z, y, ... are expressions.

2. If b is an expression, then the function abstract $(\lambda x)b$ is an expression.

3. If a and c are expressions, then c(a), the result of applying c to a, is an expression.

4. If a and b are expressions, then the ordered pair (a,b) is an expression.

5. If c and $d[x,y]$ are expressions, then $E(c,(x,y)d[x,y])$ is an expression defined by the equation

$$E((a,b),(x,y)d[x,y]) = d[a,b]$$

6. If a and b are expressions, then $i(a)$, the left injection of a, and $j(b)$, the right injection of b, are expressions.

7. If c, $d[x]$ and $e[y]$ are expressions, then $D(c,(x)d[x],(y)e[y])$ is an expression defined by the equations

$$D(i(a),(x)d[x],(y)e[y]) = d[a]$$
$$D(j(b),(x)d[x],(y)e[y]) = e[b]$$

8. r, the proof that a is identical with a, is an expression.

9. $1_n, \ldots, n_n$ are expressions for $n = 1, 2, \ldots$.

10. If c, $d_1, \ldots, d_n$ are expressions, then $R_n(c,d_1,\ldots,d_n)$ is an expression defined by the equations

$$R_n(m_n,d_1,\ldots,d_n) = d_m \qquad (m = 1, \ldots, n)$$

11. 0 is an expression, and, if a is an expression, then $s(a)$ is an expression.

12. If c, d and $e[x,y]$ are expressions, then $R(c,d,(x,y)e[x,y])$ is an expression defined by the recursion equations

$$R(0,d,(x,y)e[x,y]) = d$$
$$R(s(a),d,(x,y)e[x,y]) = e[a,R(a,d,(x,y)e[x,y])]$$

Rules of type formation.

1. If A and $B[x]$ are types, and x does not occur free in A, then

$$(\Pi x \in A)B[x]$$

is a type, namely the type of functions which take an arbitrary object a of type A into an object of type $B[a]$. When $B[x]$ does not depend on x, we abbreviate $(\Pi x \in A)B[x]$ as

$$A \rightarrow B$$

which is the type of functions from A to B.

2. If A and $B[x]$ are types, and x does not occur free in A, then

$$(\Sigma x \in A)B[x]$$

is a type, namely the type of pairs (a,b), where a is an arbitrary object in A and b an arbitrary object in $B[a]$. When $B[x]$ does not depend on x, we abbreviate $(\Sigma x \in A)B[x]$ as

$$A \times B$$

which is the cartesian product of A and B.

3. If A and B are types, then

$$A + B$$

is a type, namely the disjoint union of A and B.

4. $I(a,b)$

is a type for arbitrary expressions a and b, namely the type of proofs that a and b are identical.

5. N_n

is a type for each n = 0, 1, ..., namely the finite type with n elements.

6. N

is a type, namely the type of natural numbers.

7. V

is a type, namely the type of all objects.

<u>Axioms and rules of inference</u>.

Small letters denote expressions and capital letters denote types.

1. Definiendum = definiens.

$$((\lambda x)b[x])(a) = b[a]$$

$$E((a,b),(x,y)d[x,y]) = d[a,b]$$

$$D(i(a),(x)d[x],(y)e[y]) = d[a]$$
$$D(j(b),(x)d[x],(y)e[y]) = e[b]$$

$$R_n(m_n,d_1,\ldots,d_n) = d_m \qquad (m = 1, \ldots, n)$$

$$R(0,d,(x,y)e[x,y]) = d$$

$$R(s(a),d,(x,y)e[x,y]) = e[a,R(a,d,(x,y)e[x,y])]$$

2. Reflexivity.

$$a = a$$

3. Symmetry.

$$\frac{a = b}{b = a}$$

4. Transitivity.

$$\frac{a = b \qquad b = c}{a = c}$$

5. Preservation of equality under substitution.

$$\frac{a = b}{c[a] = c[b]}$$

6. Compatibility of epsilon with equality to the left.

$$\frac{a = b \qquad a \in C}{b \in C}$$

7. Compatibility of epsilon with equality to the right.

$$\frac{a = b \qquad c \in C[a]}{c \in C[b]}$$

8. Assumption.

$$x \in A$$

9. Π-introduction.

$$\frac{\begin{matrix}[x \in A] \\ b[x] \in B[x]\end{matrix}}{(\lambda x)b[x] \in (\Pi x \in A)B[x]}$$

Restriction: x must occur free neither in A nor in any assumption other than $x \in A$.

10. Π-elimination.

$$\frac{c \in (\Pi x \in A)B[x] \qquad a \in A}{c(a) \in B[a]}$$

11. Σ-introduction.

$$\frac{a \in A \qquad b \in B[a]}{(a,b) \in (\Sigma x \in A)B[x]}$$

12. Σ-elimination.

$$\frac{c \in (\Sigma x \in A)B[x] \qquad \begin{matrix}[x \in A] \quad [y \in B[x]] \\ d[x,y] \in C[(x,y)]\end{matrix}}{E(c,(x,y)d[x,y]) \in C[c]}$$

Restriction: x must occur free neither in A nor in any assumption on which $d[x,y] \in C[(x,y)]$ depends other than $x \in A$ and $y \in B[x]$, and y must occur free neither in B[x]

nor in any assumption on which $d[x,y] \in C[(x,y)]$ depends other than $y \in B[x]$. x and y must not occur free in $C[z]$.

13. + -introduction.

$$\frac{a \in A}{i(a) \in A + B} \qquad \frac{b \in B}{j(b) \in A + B}$$

14. + -elimination.

$$\frac{c \in A + B \qquad \begin{matrix}[x \in A] \\ d[x] \in C[i(x)]\end{matrix} \qquad \begin{matrix}[y \in B] \\ e[y] \in C[j(y)]\end{matrix}}{D(c,(x)d[x],(y)e[y]) \in C[c]}$$

Restriction: x must occur free neither in A nor in any assumption on which $d[x] \in C[i(x)]$ depends other than $x \in A$, and y must occur free neither in B nor in any assumption on which $e[y] \in C[j(y)]$ depends other than $y \in B$. x and y must not occur free in $C[z]$.

15. I -introduction.

$$r \in I(a,a)$$

16. I -elimination.

$$\frac{c \in I(a,b) \qquad d \in C[a,r]}{d \in C[b,c]}$$

17. N_n -introduction.

$$m_n \in N_n \qquad (m = 1, \ldots, n)$$

18. N_n -elimination.

$$\frac{c \in N_n \qquad d_1 \in C[1_n] \qquad \ldots \qquad d_n \in C[n_n]}{R_n(c,d_1,\ldots,d_n) \in C[c]}$$

19. N -introduction.

$$0 \in N \qquad\qquad \frac{a \in N}{s(a) \in N}$$

20. N -elimination.

$$\frac{c \in N \qquad d \in C[0] \qquad \begin{array}{c}[x \in N] \quad [y \in C[x]] \\ e[x,y] \in C[s(x)]\end{array}}{R(c,d,(x,y)e[x,y]) \in C[c]}$$

Restriction: x must not occur free in any assumption on which $e[x,y] \in C[s(x)]$ depends other than $x \in N$ and $y \in C[x]$, and y must not occur free in any assumption on which $e[x,y] \in C[s(x)]$ depends other than $y \in C[x]$. x and y must not occur free in $C[z]$.

21. V -introduction.

$a \in V$

3. LOGICAL FORMULATION OF THE THEORY OF CONSTRUCTIONS (LTC).

The theory LTC has the same expressions as TTC, but the types are replaced by formulas formulated in ordinary predicate logic. We will continue to use natural deduction and the notational conventions made in TTC.

Rules of expression formation.

The same as in TTC.

Rules of formula formation.

1. If $B[x]$ is a formula, then $(\forall x)B[x]$ and $(\exists x)B[x]$ are formulas.

2. If A and B are formulas, then $(A \supset B)$, $(A \& B)$ and $(A \vee B)$ are formulas.

3. $\perp$ is a formula.

4. $a = b$ is a formula for arbitrary expressions a and b.

5. $N(a)$ is a formula for an arbitrary expression a.

Axioms and rules of inference.

1. Definiendum = definiens.

The same as in TTC.

2. ∀-introduction.

$$\frac{B[x]}{(\forall x)B[x]}$$

Restriction: x must not occur free in any assumption on which B[x] depends.

3. ∀-elimination.

$$\frac{(\forall x)B[x]}{B[a]}$$

4. ∃-introduction.

$$\frac{B[a]}{(\exists x)B[x]}$$

5. ∃-elimination.

$$\frac{(\exists x)B[x] \qquad \begin{matrix}[B[x]] \\ C\end{matrix}}{C}$$

Restriction: x must occur free neither in C nor in any assumption on which the upper occurrence of C depends other than B[x].

6. ⊃-introduction.

$$\frac{\begin{matrix}[A] \\ B\end{matrix}}{A \supset B}$$

7. $\supset$ -elimination.

$$\frac{A \supset B \qquad A}{B}$$

8. & -introduction.

$$\frac{A \qquad B}{A \,\&\, B}$$

9. & -elimination.

$$\frac{A \,\&\, B}{A} \qquad\qquad \frac{A \,\&\, B}{B}$$

10. $\vee$ -introduction.

$$\frac{A}{A \vee B} \qquad\qquad \frac{B}{A \vee B}$$

11. $\vee$ -elimination.

$$\frac{A \vee B \qquad \begin{matrix}[A] \\ C\end{matrix} \qquad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

12. $\perp$ -elimination.

$$\frac{\perp}{C}$$

13. = -introduction.

$$a = a$$

14. = -elimination.

$$\frac{a = b \qquad C[a]}{C[b]}$$

15. N -introduction.

$$N(0) \qquad\qquad \frac{N(a)}{N(s(a))}$$

16. N -elimination.

$$\frac{N(c) \qquad C[0] \qquad \begin{array}{c}[C[x]] \\ C[s(x)]\end{array}}{C[c]}$$

Restriction: x must not occur free in any assumption on which $C[s(x)]$ depends other than $C[x]$.

17. Assumption.

A

4. INTERPRETATION OF THE LOGICAL SYSTEM IN THE TYPE THEORETIC SYSTEM.

To each formula A in LTC we associate a type A^o in TTC. A^o will have the same variables free as A. With a derivation of

$$A$$

in LTC from the assumptions

$$A_1, \ldots, A_n$$

we will associate an expression a^o such that

$$a^o \in A^o$$

is derivable in TTC from the assumptions

$$x_1^o \in A_1^o, \ldots, x_n^o \in A_n^o$$

where $x_1^o, \ldots, x_n^o$ are new variables not occuring free in $A_1^o, \ldots, A_n^o$. The free variables of a^o will be among $x_1^o, \ldots, x_n^o$ and the variables which occur free in $A_1^o, \ldots, A_n^o$ and A^o. The substitution property

$$A[a]^o = A^o[a]$$

will also be satisfied, where the equality means syntactical equality.

The definition of A^o is by induction on the formation of the formula A in LTC.

$(\forall x)B[x]$ is interpreted as the type of functions which to each object a of the universe give you a proof of $B[a]$, that is, an object of type $B^{o}[a]$:

$$((\forall x)B[x])^{o} \equiv (\Pi x \in V)B^{o}[x]$$

$(\exists x)B[x]$ is interpreted as the type of pairs (a,b), where a is an object of the universe and b a proof of $B[a]$, that is, an object of type $B^{o}[a]$:

$$((\exists x)B[x])^{o} \equiv (\Sigma x \in V)B^{o}[x]$$

$A \supset B$ is interpreted as the type of functions which to each proof of A, that is, object of type A^{o}, give you a proof of B, that is, an object of type B^{o}:

$$(A \supset B)^{o} \equiv A^{o} \rightarrow B^{o}$$

A & B is interpreted as the type of pairs (a,b) where a is a proof of A, that is, an object of type A^{o}, and b is a proof of B, that is, an object of type B^{o}:

$$(A \mathbin{\&} B)^{o} \equiv A^{o} \times B^{o}$$

$A \vee B$ is interpreted as the type, an object of which is either a proof of A, together with the information that it is the left disjunct that it is a proof of, or a proof of B, together with the information that it is the right disjunct that it is a proof of. Thus $A \vee B$ is interpreted as the disjoint union of A^{o} and B^{o}:

$$(A \vee B)^{\circ} \equiv A^{\circ} + B^{\circ}$$

$\perp$, the absurdity, is not possible to prove and is therefore interpreted as the empty type:

$$\perp^{\circ} \equiv N_0$$

a = b is interpreted as the type of proofs that a is identical with b:

$$(a = b)^{\circ} \equiv I(a,b)$$

N(a) is interpreted as the type of pairs (b,c), where b is a natural number and c is a proof of the identity of a and b:

$$N(a)^{\circ} \equiv (\Sigma x \in N)I(a,x)$$

This finishes the construction of A°.

By induction on the construction of A°, it is now obvious that the above-mentioned substitution property

$$A[a]^{\circ} = A^{\circ}[a]$$

is satisfied.

We will now show that if A is derivable from $A_1, \ldots, A_n$ in LTC, then we can construct an expression a° and give a derivation of $a^{\circ} \in A^{\circ}$ from $x_1^{\circ} \in A_1^{\circ}, \ldots, x_n^{\circ} \in A_n^{\circ}$ in TTC. The proof will be by induction on the length of the derivation in LTC. In the proof we will not, for notational convenience, show explicitly any assumptions other than those that are directly involved in the inference under consideration.

It now remains to go through all the axioms and rules of inference in LTC.

1. Definiendum = definiens.

We want to construct an expression c^{o} such that

$$c^{o} \in (a = b)^{o}$$

when a is a definiendum and b the corresponding definiens. Since the definiendum = definiens axioms are the same in LTC and TTC we have in TTC

$$a = b$$

By I -introduction we also have

$$r \in I(a,a)$$

We can now use compatibility of epsilon with equality to the right to get

$$r \in I(a,b)$$

Since

$$(a = b)^{o} \equiv I(a,b)$$

we have

$$r \in (a = b)^{o}$$

so that we can choose c^{o} to be r.

2. $\forall$-introduction.

$$\frac{B[x]}{(\forall x)B[x]}$$

According to the induction hypothesis we have constructed an expression $b^{o}[x]$ and a derivation of

$$b^{o}[x] \in B^{o}[x]$$

Hence we can apply Π-introduction and get

$$(\lambda x)b^{o}[x] \in (\Pi x \in V)B^{o}[x]$$

that is, by the definition of $((\forall x)B[x])^{o}$,

$$(\lambda x)b^{o}[x] \in ((\forall x)B[x])^{o}$$

as desired. Note that the restriction on the assumptions in the derivation of $B[x]$, which must be satisfied in order to apply $\forall$-introduction, immediatly gives the right restriction on the assumptions in the derivation of $b^{o}[x] \in B^{o}[x]$ in order to apply Π-introduction.

3. $\forall$-elimination.

$$\frac{(\forall x)B[x]}{B[a]}$$

By the induction hypothesis we have constructed an expression c^{o} and a derivation of

$$c^{o} \in (\Pi x \in V)B^{o}[x]$$

V -introduction gives

$$a \in V$$

and Π-elimination

$$c^{o}(a) \in B^{o}[a]$$

With the substitution property

$$B[a]^{o} = B^{o}[a]$$

we get

$$c^{o}(a) \in B[a]^{o}$$

as desired.

4. $\exists$ -introduction.

$$\frac{B[a]}{(\exists x)B[x]}$$

We have an expression b^{o} and a derivation of

$$b^{o} \in B^{o}[a]$$

according to the induction hypothesis and the substitution property. V -introduction and Σ-introduction give

$$(a,b^{o}) \in (\Sigma x \in V)B^{o}[x]$$

Hence, by the definition of $((\exists x)B[x])^{o}$,

$$(a,b^{o}) \in ((\exists x)B[x])^{o}$$

as desired.

5. $\exists$-elimination.

$$\frac{(\exists x)B[x] \qquad \begin{matrix}[B[x]] \\ C\end{matrix}}{C}$$

According to the induction hypothesis we have constructed an expression c^{o} and a derivation of

$$c^{o} \in (\Sigma x \in V)B^{o}[x]$$

We have also constructed an expression $d^{o}[x,y^{o}]$ and a derivation of

$$d[x,y^{o}] \in C^{o}$$

from the assumption

$$y^{o} \in B[x]^{o}$$

The restriction on the assumptions in the derivation of C from B[x] makes it possible to apply Σ-elimination and get

$$E(c^{o},(x,y^{o})d^{o}[x,y^{o}]) \in C^{o}$$

thereby discharging the assumption $y^{o} \in B[x]^{o}$.

6. $\supset$-introduction.

$$\frac{\begin{matrix}[A] \\ B\end{matrix}}{A \supset B}$$

By the induction hypothesis we have constructed a term b^{o}

and a derivation of

$$b^{o}[x^{o}] \in B^{o}$$

from the assumption

$$x^{o} \in A^{o}$$

We can then use Π-introduction and get

$$(\lambda x^{o})b^{o}[x^{o}] \in (\Pi x^{o} \in A^{o})B^{o}$$

thereby discharging the assumption $x^{o} \in A^{o}$. Since x^{o} does not occur free in B^{o} we have

$$(\lambda x^{o})b^{o}[x^{o}] \in A^{o} \rightarrow B^{o}$$

Hence, by the definition of $(A \supset B)^{o}$,

$$(\lambda x^{o})b^{o}[x^{o}] \in (A \supset B)^{o}$$

as desired.

7. $\supset$-elimination.

$$\frac{A \supset B \qquad A}{B}$$

According to the induction hypothesis we have an expression c^{o} and a derivation of

$$c^{o} \in A^{o} \rightarrow B^{o}$$

and an expression a^{o} and a derivation of

$$a^{o} \in A^{o}$$

We can then apply Π-elimination and get

$$c^o(a^o) \in B^o$$

as desired.

8. & -introduction.

$$\frac{A \qquad B}{A \,\&\, B}$$

By the induction hypothesis we have expressions a^o and b^o and derivations of

$$a^o \in A^o$$

and

$$b^o \in B^o$$

Σ -introduction gives

$$(a^o, b^o) \in A^o \times B^o$$

that is, by the definition of $(A \,\&\, B)^o$,

$$(a^o, b^o) \in (A \,\&\, B)^o$$

as desired.

9. & -elimination.

$$\frac{A \,\&\, B}{A} \qquad\qquad \frac{A \,\&\, B}{B}$$

By the induction hypothesis we have an expression c^o and a

derivation of

$$c^{o} \in A^{o} \times B^{o}$$

Σ-elimination then gives

$$E(c^{o},(x,y)x) \in A^{o}$$

and

$$E(c^{o},(x,y)y) \in B^{o}$$

as desired.

10. $\vee$-introduction.

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

According to the induction hypothesis we have expressions a^{o} and b^{o} and derivations of

$$a^{o} \in A^{o}$$

and

$$b^{o} \in B^{o}$$

respectively. We can then use + -introduction to get

$$i(a^{o}) \in A^{o} + B^{o}$$

and

$$j(b^{o}) \in A^{o} + B^{o}$$

respectively.

By the definition of $(A \vee B)^o$ we then have

$$i(a^o) \in (A \vee B)^o$$

and

$$j(b^o) \in (A \vee B)^o$$

as desired.

11. $\vee$-elimination.

$$\frac{A \vee B \qquad \begin{matrix}[A] \\ C\end{matrix} \qquad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

According to the induction hypothesis we have constructed expressions c^o, $d^o[x^o]$ and $e^o[y^o]$ and derivations of

$$c^o \in A^o + B^o$$

and

$$d^o[x^o] \in C^o$$

from the assumption

$$x^o \in A^o$$

and

$$e^o[y^o] \in C^o$$

from the assumption

$$y^o \in B^o$$

We can now use + -elimination to get

$$D(c^{o},(x^{o})d^{o}[x^{o}],(y^{o})e^{o}[y^{o}]) \in C^{o}$$

thereby discharging the assumptions $x^{o} \in A^{o}$ and $y^{o} \in B^{o}$.

12. $\perp$-elimination.

$$\frac{\perp}{C}$$

By the induction hypothesis we have constructed an expression a^{o} and a derivation of

$$a^{o} \in N_0$$

N_0 -elimination then gives

$$R_0(a^{o}) \in C^{o}$$

as desired.

13. = -introduction.

$$a = a$$

By I -introduction we have

$$r \in I(a,a)$$

as desired.

14. = -elimination.

$$\frac{a = b \qquad C[a]}{C[b]}$$

According to the induction hypothesis we have expressions c^{o} and d^{o} and derivations of

$$c^{o} \in I(a,b) \qquad (1)$$

and

$$d^{o} \in C[a]^{o} \qquad (2)$$

By the substitution property we have

$$C[a]^{o} = C^{o}[a] \qquad (3)$$

By using (3) we can apply I -elimination to (1) and (2) to get

$$d^{o} \in C^{o}[b]$$

The substitution property used once more finally gives

$$d^{o} \in C[b]^{o}$$

as desired.

In the cases below dealing with the rules for N, we will use p(c) and q(c) as abbreviations for E(c,(x,y)x) and E(c,(x,y)y) respectively. If

$$c \in (\sum x \in A)B[x]$$

we have, by $\sum$-elimination,

$$p(c) \in A$$

and

$$q(c) \in B[p(c)]$$

15. N -introduction.

$$N(0)$$

By N -introduction, I -introduction and Σ-introduction we have

$$(0,r) \in (\Sigma x \in N)I(0,x)$$

that is, by the definition of $N(0)^{o}$,

$$(0,r) \in N(0)^{o}$$

as desired.

$$\frac{N(a)}{N(s(a))}$$

According to the induction hypothesis we have an expression b^{o} and a derivation of

$$b^{o} \in (\Sigma x \in N)I(a,x) \quad (1)$$

From (1) and the above rules for p and q we get

$$p(b^{o}) \in N \quad (2)$$

and

$$q(b^{o}) \in I(a,p(b^{o})) \quad (3)$$

N -introduction and (2) give

$$s(p(b^{o})) \in N \quad (4)$$

(4), I -introduction and Σ -introduction give

$$(s(p(b^o)),r) \in (\textstyle\sum x \in N)I(s(p(b^o)),x) \qquad (5)$$

By using (3), (5) and I -elimination we finally get

$$(s(p(b^o)),r) \in (\textstyle\sum x \in N)I(s(a),x)$$

as desired.

16. N -elimination.

$$\frac{N(c) \qquad C[0] \qquad \begin{matrix}[C[x]] \\ C[s(x)]\end{matrix}}{C[c]}$$

According to the induction hypothesis we have expressions b^o and d^o and derivations of

$$b^o \in (\textstyle\sum x \in N)I(c,x) \qquad (1)$$

and

$$d^o \in C^o[0] \qquad (2)$$

where in (2) we have used the substitution property. We also have an expression $e^o[x,y^o]$ and, by using the substitution property again, a derivation of

$$e^o[x,y^o] \in C^o[s(x)] \qquad (3)$$

from the assumption

$$y^o \in C^o[x] \qquad (4)$$

From (1) we get

$$p(b^o) \in N \qquad (5)$$

and

$$q(b^o) \in I(c,p(b^o)) \qquad (6)$$

We can now apply N -elimination to (5), (2) and the derivation of (3) from (4) to get

$$R(p(b^o),d^o,(x,y^o)e^o[x,y^o]) \in C^o[p(b^o)] \qquad (7)$$

(6), (7) and I -elimination together with the substitution property finally give

$$R(p(b^o),d^o,(x,y^o)e^o[x,y^o]) \in C[c]^o$$

as desired.

17. Assumption.

A

This corresponds to introducing a variable x^o and making the assumption

$$x^o \in A^o$$

This was the final step in the proof.

5. INTERPRETATION OF THE TYPE THEORETIC SYSTEM IN THE LOGICAL SYSTEM.

To each type A in TTC we associate a formula $A^*[z]$ in LTC. $A^*[z]$ will have as free variables the variables free in A and a new variable z. We will show that if

$$a \in A$$

is derivable in TTC from the assumptions

$$x_1 \in A_1, \ldots, x_n \in A_n$$

then

$$A^*[a]$$

is derivable in LTC from the assumptions

$$A_1^*[x_1], \ldots, A_n^*[x_n]$$

The substitution property

$$A[a]^*[z] = A^*[a,z]$$

will also be satisfied, where the equality means syntactical equality.

$A^*[z]$ will be constructed by induction on its formation according to the rules of type formation in TTC.

$(\prod x \in A)B[x]$ is interpreted as the species of objects c such that, for all elements a of the species $A^*[x]$, c(a)

is an element of the species $B^*[a,y]$:

$$((\prod x \in A)B[x])^*[z] \equiv (\forall x)(A^*[x] \supset B^*[x,z(x)])$$

$(\sum x \in A)B[x]$ is interpreted as the species of pairs (a,b) such that, a is an element of the species $A^*[x]$ and b is an element of the species $B^*[a,y]$:

$$((\sum x \in A)B[x])^*[z] \equiv$$
$$(\exists x)(\exists y)(A^*[x] \ \& \ B^*[x,y] \ \& \ z = (x,y))$$

A + B is interpreted as the disjoint union of the species $A^*[x]$ and $B^*[y]$:

$$(A + B)^*[z] \equiv$$
$$(\exists x)(A^*[x] \ \& \ z = i(x)) \vee (\exists y)(B^*[y] \ \& \ z = j(y))$$

I(a,b) is interpreted as the species which contains the single element r provided a = b:

$$I(a,b)^*[z] \equiv (a = b \ \& \ z = r)$$

N_o is interpreted as absurdity:

$$N_o^*[z] \equiv \perp$$

N_n, for n = 1, 2, ..., is interpreted as the species whose elements are 1_n, ..., n_n:

$$N_n^*[z] \equiv (z = 1_n \vee \dots \vee z = n_n)$$

N is interpreted as the species of natural numbers:

$$N^*[z] \equiv N(z)$$

V is interpreted as the species of all objects:

$$V^*[z] \equiv (z = z)$$

It only remains to interpret a = b and that is simply done by putting

$$(a = b)^* \equiv (a = b)$$

It is now easy to see, by induction on the construction of $A^*[z]$, that the above-mentioned substitution property

$$A[a]^*[z] = A^*[a,z]$$

is satisfied.

We will now show that if $a \in A$ is derivable from $x_1 \in A_1, \ldots, x_n \in A_n$ in TTC, then $A^*[a]$ is derivable from $A_1^*[x_1], \ldots, A_n^*[x_n]$ in LTC. The proof will be by induction on the length of the derivation in TTC. In the proof we will not, for notational convenience, show explicitly any assumptions other than those that are directly involved in the inference under consideration. It now remains to go through all the axioms and rules of inference of TTC.

1 - 5. We have to show that if a = b is derivable in TTC, then it is also derivable in LTC. The cases of definiendum = definiens and reflexivity are trivial, and the cases of symmetry, transitivity and preservation of equality under substitution are easily shown by using = -introduction and = -elimination.

6. Compatibility of epsilon with equality to the left.

$$\frac{a = b \qquad a \in C}{b \in C}$$

By the induction hypothesis we have a derivation in LTC of

$$C^{*}[a]$$

We have already noted above that if

$$a = b$$

is derivable in TTC, then it is also derivable in LTC. We can then apply = -elimination to get

$$C^{*}[b]$$

as desired.

7. Compatibility of epsilon with equality to the right.

$$\frac{a = b \qquad c \in C[a]}{c \in C[b]}$$

According to the induction hypothesis we have a derivation of

$$C[a]^{*}[c]$$

The substitution property gives

$$C[a]^{*}[c] = C^{*}[a,c]$$

and since we have, as noted above, a derivation of

$$a = b$$

in LTC, we can apply = -elimination to get

$$C^*[b, c]$$

The substitution property used once more gives

$$C[b]^*[c]$$

as desired.

8. Assumption.

$$x \in A$$

corresponds to the assumption of

$$A^*[x]$$

9. Π -introduction.

$$\frac{\begin{array}{c}[x \in A] \\ b[x] \in B[x]\end{array}}{(\lambda x)b[x] \in (\Pi x \in A)B[x]}$$

By the induction hypothesis we have a derivation of

$$B[x]^*[b[x]]$$

from the assumption

$$A^*[x]$$

By using $\supset$ -introduction we get a derivation of

$$A^*[x] \supset B^*[x, b[x]] \qquad (1)$$

thereby discharging the assumption $A^*[x]$. By definiendum = definiens we have

$$((\lambda x)b[x])(x) = b[x] \qquad (2)$$

(1), (2) and = -elimination give

$$A^*[x] \supset B^*[x, ((\lambda x)b[x])(x)] \qquad (3)$$

Since the restriction on the assumptions in the derivation of $b[x] \in B[x]$ from $x \in A$ gives the right restriction on the assumptions in the derivation of (3) in order to apply $\forall$ -introduction, we get

$$(\forall x)(A^*[x] \supset B^*[x, ((\lambda x)b[x])(x)])$$

and we have then, by the definition of $((\Pi x \in A)B[x])^*[z]$, a derivation of

$$((\Pi x \in A)B[x])^*[(\lambda x)b[x]]$$

as desired.

10. Π -elimination.

$$\frac{c \in (\Pi x \in A)B[x] \qquad a \in A}{c(a) \in B[a]}$$

By the induction hypothesis we have derivations of

$$(\forall x)(A^*[x] \supset B^*[x, c(x)]) \qquad (1)$$

and

$$A^*[a] \qquad (2)$$

$\forall$-elimination applied to (1) gives

$$A^*[a] \supset B^*[a, c(a)] \qquad (3)$$

(2), (3), $\supset$-elimination and the substitution property finally give

$$B[a]^*[c(a)]$$

as desired.

11. Σ-introduction.

$$\frac{a \in A \qquad b \in B[a]}{(a,b) \in (\Sigma x \in A)B[x]}$$

By the induction hypothesis we have derivations of

$$A^*[a]$$

and

$$B[a]^*[b]$$

which give, together with the substitution property and & -introduction,

$$A^*[a] \ \& \ B^*[a,b]$$

= -introduction and & -introduction then give

$$A^*[a] \ \& \ B^*[a,b] \ \& \ (a,b) = (a,b)$$

By using $\exists$-introduction twice we get

$$(\exists x)(\exists y)(A^*[x] \ \& \ B^*[x,y] \ \& \ (a,b) = (x,y))$$

that is, by the definition of $((\sum x \in A)B[x])^*[z]$,

$$((\sum x \in A)B[x])^*[(a,b)]$$

as desired.

12. $\sum$ -elimination.

$$\frac{c \in (\sum x \in A)B[x] \qquad \begin{array}{c}[x \in A] \quad [y \in B[x]] \\ d[x,y] \in C[(x,y)]\end{array}}{E(c,(x,y)d[x,y]) \in C[c]}$$

By the induction hypothesis we have a derivation of

$$(\exists x)(\exists y)(A^*[x] \ \& \ B^*[x,y] \ \& \ c = (x,y)) \qquad (1)$$

and we also have a derivation of

$$C[(x,y)]^*[d[x,y]]$$

from

$$A^*[x]$$

and

$$B[x]^*[y]$$

We make the assumption

$$A^*[x] \ \& \ B^*[x,y] \ \& \ c = (x,y) \qquad (2)$$

and we can then from (2) derive

$$C[(x,y)]^*[d[x,y]] \qquad (3)$$

By definiendum = definiens we have

$$E((x,y),(x,y)d[x,y]) = d[x,y] \tag{4}$$

(3), (4) and = -elimination give

$$C[(x,y)]^{*}[E((x,y),(x,y)d[x,y])] \tag{5}$$

From (2) we also get

$$c = (x,y) \tag{6}$$

(5), (6), the substitution property and = -elimination give

$$C[c]^{*}[E(c,(x,y)d[x,y])] \tag{7}$$

Finally, we use $\exists$-elimination twice to get a derivation of (7), thereby discharging the assumption (2), which is permissible since the right restriction on the assumptions in the derivation of (7) is satisfied.

13. + -introduction.

$$\frac{a \in A}{i(a) \in A + B} \qquad \frac{b \in B}{j(b) \in A + B}$$

By the induction hypothesis we have a derivation of

$$A^{*}[a]$$

= -introduction and & -introduction then give

$$A^{*}[a] \;\&\; i(a) = i(a)$$

and by $\exists$-introduction and $\vee$-introduction we get

$$(\exists x)(A^{*}[x] \;\&\; i(a) = i(x)) \vee (\exists y)(B^{*}[y] \;\&\; i(a) = j(y))$$

that is, by the definition of $(A + B)^*[z]$,

$$(A + B)^*[i(a)]$$

as desired. The second case is handled in the same way.

14. + -elimination.

$$\frac{c \in A + B \qquad \begin{matrix}[x \in A] \\ d[x] \in C[i(x)]\end{matrix} \qquad \begin{matrix}[y \in B] \\ e[y] \in C[j(y)]\end{matrix}}{D(c,(x)d[x],(y)e[y]) \in C[c]}$$

We assume

$$A^*[x] \;\&\; c = i(x) \tag{1}$$

From (1) we get, by & -elimination,

$$A^*[x] \tag{2}$$

and from (2) we can derive, by the induction hypothesis,

$$C[i(x)]^*[d[x]] \tag{3}$$

By definiendum = definiens we have

$$D(i(x),(x)d[x],(y)e[y]) = d[x] \tag{4}$$

(3), (4) and = -elimination give

$$C[i(x)]^*[D(i(x),(x)d[x],(y)e[y])] \tag{5}$$

From (1) we get, by & -elimination,

$$c = i(x) \tag{6}$$

(5), (6) and = -elimination give, together with the

substitution property used twice,

$$C[c]^*[D(c,(x)d[x],(y)e[y])] \qquad (7)$$

We can now use $\exists$ -elimination to discharge the assumption (1) and get a derivation of (7) from

$$(\exists x)(A^*[x] \;\&\; c = i(x)) \qquad (8)$$

By a similar derivation we get (7) from

$$(\exists y)(B^*[y] \;\&\; c = j(y)) \qquad (9)$$

According to the induction hypothesis we have a derivation of

$$(\exists x)(A^*[x] \;\&\; c = i(x)) \vee (\exists y)(B^*[y] \;\&\; c = j(y))$$

and by using $\vee$-elimination we can discharge (8) and (9) and get a derivation of (7) as desired.

15. I -introduction.

$$r \in I(a,a)$$

By = -introduction and & -introduction we have

$$a = a \;\&\; r = r$$

as desired.

16. I -elimination.

$$\frac{c \in I(a,b) \qquad d \in C[a,r]}{d \in C[b,c]}$$

By the induction hypothesis we have derivations of

$$a = b \,\&\, c = r$$

and

$$C[a,r]^{*}[d]$$

By the substitution property, & -elimination and = -elimination we get

$$C[b,c]^{*}[d]$$

as desired.

17. N_n -introduction.

$$m_n \in N_n \qquad (m = 1, \ldots, n)$$

From

$$m_n = m_n$$

which is an instance of = -introduction, we get, by repeated use of $\vee$-introduction,

$$m_n = 1_n \vee \ldots \vee m_n = n_n$$

that is, by the definition of $N_n^{*}[z]$,

$$N_n^{*}[m_n]$$

as desired.

18. N_n -elimination.

$n = 0$.

$$\frac{c \in N_0}{R_0(c) \in C[c]}$$

By the induction hypothesis we have a derivation of

$$\bot$$

so that we can use $\bot$ -elimination to get

$$C[c]^*[R_0(c)]$$

as desired.

$n = 1, 2, \ldots$

$$\frac{c \in N_n \qquad d_1 \in C[1_n] \quad \ldots \quad d_n \in C[n_n]}{R_n(c, d_1, \ldots, d_n) \in C[c]}$$

According to the induction hypothesis we have derivations of

$$C[m_n]^*[d_m] \tag{m}$$

for $m = 1, \ldots, n$. We make the assumptions

$$c = m_n \tag{m'}$$

By definiendum = definiens we have

$$R_n(m_n, d_1, \ldots, d_n) = d_m \tag{m"}$$

For each m, = -elimination, (m), (m') and (m") give,

together with the substitution property,

$$C[c]^{*}[R_n(c,d_1,\dots,d_n)] \qquad (n+1)$$

By the induction hypothesis we have a derivation of

$$c = 1_n \vee \dots \vee c = n_n$$

so that, by repeated use of $\vee$-elimination, we can discharge the assumptions (m'), m = 1, ..., n, and get a derivation of (n+1) as desired.

19. N -introduction.

$$0 \in N$$

The definition of $N^{*}[z]$ and N -introduction immediately gives

$$N^{*}[0]$$

as desired.

$$\frac{a \in N}{s(a) \in N}$$

According to the induction hypothesis we have a derivation of

$$N^{*}[a]$$

The definition of $N^{*}[z]$ and N -introduction then gives

$$N^{*}[s(a)]$$

as desired.

20. N -elimination.

$$\frac{c \in N \qquad d \in C[0] \qquad \begin{matrix}[x \in N] & [y \in C[x]] \\ e[x,y] \in C[s(x)] & \end{matrix}}{R(c,d,(x,y)e[x,y]) \in C[c]}$$

By the induction hypothesis we have a derivation of

$$C[s(x)]^{*}[e[x,y]] \tag{1}$$

from the assumptions

$$N^{*}[x] \tag{2}$$

and

$$C[x]^{*}[y] \tag{3}$$

By applying $\supset$ -introduction to (1) we get a derivation of

$$C[x]^{*}[y] \supset C[s(x)]^{*}[e[x,y]] \tag{4}$$

thereby discharging the assumption (3). We apply $\forall$-introduction to (4) to get

$$(\forall y)(C[x]^{*}[y] \supset C[s(x)]^{*}[e[x,y]]) \tag{5}$$

From (5) we get, by $\forall$ -elimination,

$$\begin{aligned} &C[x]^{*}[R(x,d,(x,y)e[x,y])] \supset \\ &C[s(x)]^{*}[e[x,R(x,d,(x,y)e[x,y])]] \end{aligned} \tag{6}$$

We assume

$$C[x]^{*}[R(x,d,(x,y)e[x,y])] \tag{7}$$

(6), (7) and $\supset$ -elimination give

$$C[s(x)]^{*}[e[x,R(x,(x,y)e[x,y])]] \quad (8)$$

By definiendum = definiens we have

$$R(s(x),d,(x,y)e[x,y]) = e[x,R(x,d,(x,y)e[x,y])] \quad (9)$$

(8), (9) and = -elimination give

$$C[s(x)]^{*}[R(s(x),d,(x,y)e[x,y])] \quad (10)$$

From (2) we get, by N -introduction,

$$N^{*}[s(x)] \quad (11)$$

(10), (11) and & -introduction give

$$N^{*}[s(x)] \;\&\; C[s(x)]^{*}[R(s(x),d,(x,y)e[x,y])] \quad (12)$$

We now have a derivation of (12) from (2) and (7). We therefore can derive (12) from the assumption

$$N^{*}[x] \;\&\; C[x]^{*}[R(x,d,(x,y)e[x,y])] \quad (13)$$

By the induction hypothesis we also have derivations of

$$N(c) \quad (14)$$

and

$$C[0]^{*}[d] \quad (15)$$

By definiendum = definiens we have

$$R(0,d,(x,y)e[x,y]) = d \quad (16)$$

(15), (16) and $=$ -elimination give

$$C[0]^*[R(0,d,(x,y)e[x,y])] \tag{17}$$

and from (17) we get, by applying N -introduction and & -introduction,

$$N^*[0] \;\&\; C[0]^*[R(0,d,(x,y)e[x,y])] \tag{18}$$

Since we have derivations of (14) and (18) and we have derived (12) from (13) we can apply N -elimination and the substitution property to get

$$N^*[c] \;\&\; C[c]^*[R(c,d,(x,y)e[x,y])] \tag{19}$$

thereby discharging the assumption (13). From (19) we finally get, by applying & -elimination,

$$C[c]^*[R(c,d,(x,y)e[x,y])]$$

as desired.

21. V -introduction.

$$a \in V$$

This is interpreted as

$$a = a$$

which is an instance of $=$ -introduction.

The interpretation of TTC in LTC is now complete.

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